Homework 2 Clarification

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I would like to clarify a few common problems with problems 2.3.1d and 2.3.5. Homework problem 2.3.1d states the following:

Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Prove or disprove that f is injective.

Many of the attempted solutions I read went more or less as follows:

Proof attempt. Suppose we have $x, y \in \mathbb{R}$ such that $f(x) = f(y)$, so $x^3 = y^3$. Taking the cube root of both sides, we see $\sqrt[3]{x^3} = \sqrt[3]{y^3}$, and therefore $x = y$. Thus, f is injective. □

The necessary implication to prove is correctly identified: we need to show that $x^3 = y^3 \implies x = y$. However, we *cannot* simply take the rube root of both sides, as doing so makes our argument circular. The symbol $\sqrt[n]{y}$ means '*choose* x such that $x^n = y$', but it is not a priori clear that such a choice of x is uniquely or even canonically defined. In other words, it is not obvious from the definition of $\sqrt[n]{\ }$ that $x \mapsto \sqrt[n]{x}$ even matches the definition of a function (see activity 3.5).

For instance, consider $n = 2$. Given that our domain is $\mathbb{R}$, $\sqrt{-1}$ is not defined at all, and $\sqrt{1} = \pm 1$ is only defined *up to a choice* in sign. Of course, for square roots, we know that there is a canonical choice of sign: take the positive root. But to see why abusing the $\sqrt[n]{\ }$ symbol gives us trouble, consider the following "proof" that $x \mapsto x^2$ is injective:

"Proof." Suppose we have $x, y \in \mathbb{R}$ such that $x^2 = y^2$. Taking the square root of both sides, we see $\sqrt{y^2} = \sqrt{x^2}$, so $x = y$. Thus, $x \mapsto x^2$ is injective. □

We see the flaw: $\sqrt{\ }$ is not a function since it gives two outputs for positive inputs. Equivalently, x^2 maps two inputs to the same output, such as $x = \pm 1$. The fact that the same argument with $\sqrt[3]{\ }$ gives the right answer is a coincidence, and if we want to perform operations like taking the cube root of both sides, we ought to make sure that we're not making a choice (i.e. sign) that is destroyed upon cubing. Otherwise, we would be able to find x and y such that $x^3 = y^3$ and our *choice* for $\sqrt[3]{x^3}$ equals our *choice* for $\sqrt[3]{y^3}$, but $x \neq y$. To rephrase our analogy to $x \mapsto x^2$ in these terms, we have $(-1)^2 = 1^2$ and $\sqrt{(-1)^2} = \sqrt{1^2}$, but $1 \neq -1$.

Hopefully this clarifies why arguments like the one suggested above are circular: they assume that $\sqrt[3]{\ }$ is well defined, but that assumption is basically the content of the statement that $x \mapsto x^3$ is injective. That aside, let us consider some noncircular proof strategies that $x \mapsto x^3$ is indeed injective should you wish to correct your solution.

- Try the contrapositive $x \neq y \implies x^3 \neq y^3$. (*Hint: WLOG, let $y > x$ and write $y = x + c$ for some $c > 0$.*)

- Granted that the function $x^3 - c$ can have at most 3 complex roots (we will prove this later by induction), write out 3 roots in terms of each other and show that exactly 1 is real. (*Hint: You will not be able to write the roots explicitly. Look at Corollary 2.11.*)

- Prove that $x \mapsto x^3$ is strictly increasing and proceed from there.

Exercise 2.3.5 seems to have caused people similar problems. Colloquially, the problem asks us to prove that we can define an inverse g to a bijective function $f : X \to Y$ and that g satisfies the definition of a function that we developed in activity 3.5. Several people wrote an argument akin to 'let $g = f^{-1}$', which is likely because I did not explain my intentions for the problem clearly. I intended for you to *construct* a function g (to do which you will have to use the injectivity and surjectivity of f) that has the listed properties (i.e. g is indeed f^{-1}). Of course, I could have given g the name f^{-1}, but the point is that we want to show that f^{-1} is well defined without assuming so upfront. As such, writing $g = f^{-1}$ just assumes what we are trying to prove.

If you are interested in correcting your solution to this problem, think of a function $g : Y \to X$ as a rule assigning *exactly one* element of x to *every* element of Y. Because f is bijective, f is *injective* and *surjective*. Use your mathematical gluing skills to figure out how to put the italicized words in this paragraph together precisely and prove what you want to prove.